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COMMENT

Metastability in the (1+1)D Ising model: a primitive droplet model calculation

C K Harris†

Department of Physics, University of Edinburgh, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, UK

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Abstract. Using a primitive droplet model in conjugation with recent exact results for the surface free energies of Ising droplets on a square lattice with anisotropic couplings, we obtain an expression for the principal non-universal amplitude appearing in the imaginary part of the small field metastable state energy of the Hamiltonian field theory version of the Ising model in (1+1)D. The calculation is in very good numerical agreement with the results of a recent finite lattice study.

The question of whether the free energy $f(H)$ of an Ising system is singular along the phase coexistence line $H=0$, $T<T_c$, and the nature of this possible singularity, is an old one which has been of considerable interest. It is now widely believed that the analytic continuation of $f(H)$ from real positive H into the complex H plane exhibits a branch point singularity at $H=0$, such that for small $|H|$ (Langer 1967, Gunther *et al* 1980),

$$\mathrm{Re}\, f(\mathrm{e}^{\mathrm{i}\pi}|H|) = \mathrm{Re}\, f(\mathrm{e}^{-\mathrm{i}\pi}|H|), \tag{1a}$$

$$\mathrm{Im}\, f(\mathrm{e}^{\pm\mathrm{i}\pi}|H|) = \mp B|H|^{b} \exp\{-A|H|^{-a}[1+\mathrm{O}(H^2)]\}, \tag{1b}$$

A, B are non-universal constants, while a, b are believed universal, with values

$$\begin{aligned} &a=d-1, \qquad b=\tfrac{1}{2}(3-d)d, \qquad 1<d<5 \qquad d\neq 3 \\ &b=-\tfrac{7}{3}, \qquad d=3. \end{aligned} \tag{1c}$$

The singularity structure (1) can be used to obtain high-order estimates for the expansion of $f(H)$ and its derivatives in powers of H (Gunther *et al* 1980). Such expansions, of the magnetisation, were derived from earlier activity series at various temperatures below T_c and for several lattices in two and three dimensions by Baker and Kim (1980). Although the expansions obtained for the 3D lattices were too short for their high-order behaviour to be ascertained, Baker and Kim were able to obtain the asymptotic form of the coefficients for the square and triangular lattices and deduce that $f(H)$ is singular at $H=0$. A subsequent analysis of Baker and Kim's published coefficients for the square lattice (Lowe and Wallace 1980) showed that their asymptotic behaviour was in good agreement with that expected from the singularity structure (1).

Rather precise estimates of the constants A and B can be extracted from the square lattice series results but these quantities are non-universal and are, therefore, not

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predicted by the field theoretic treatment which leads to (1). However, a classical lattice droplet theory leads to a singularity structure for $f(H)$ of the form (1) (Langer 1967) with an expression for A in terms of the surface free energy of an equilibrium Ising droplet. This result for A is expected to be correct at low temperatures where the non-interacting cluster picture on which the classical droplet theory is based is valid, though the theory predicts incorrect values for B and b owing to the neglect of droplet wobble fluctuations (which are included in the field theoretic approach). Using recent exact work on the surface free energies of square lattice Ising droplets numerical values of A were calculated for comparison with the series estimates by the present author (Harris 1984—hereafter referred to as I). A simple modification of the classical droplet theory, incorporating the effect of droplet nesting, was employed. The results were in excellent agreement with the series estimates, even for T near T_c, thus furnishing strong support for the droplet theory of phase coexistence.

The purpose of this comment is to present the extension of the droplet theory of I to the case of anisotropic couplings, and hence, after appropriate limits have been taken, to the Hamiltonian field theory version of the Ising model in (1+1)D. A direct numerical comparison with the results of a recent finite lattice study of the latter system (Hamer 1983) is then made. Our starting point is a square lattice Ising model with a reduced field H and couplings K_1 and K_2 in the x and y directions respectively. Defining the following quantities

$$\mu^{-1} = K_2\,e^{2K_1} \qquad h = H/K_2 \tag{2}$$

and taking the limit $K_1 \to \infty$, K_2, $H \to 0$ with μ, h remaining finite and arbitrary it can be shown that the 2D Ising problem is equivalent to the following 1D quantum mechanical problem (Fradkin and Susskind 1978)

$$\mathscr{H} = \text{constant} - \sum [\sigma_3(n)\sigma_3(n+1) + \mu\sigma_1(n) + h\sigma_3(n)]. \tag{3}$$

The equilibrium state and free energy in the statistical system correspond respectively to the ground state and ground state energy in the quantum system. The system described by the Hamiltonian (3) has a unique ground state for μ greater than the critical coupling of unity and a pair of degenerate ground states for $h = 0$, $\mu < 1$. For $h > 0$ (say), the unique ground state becomes metastable on reversing the sign of h. Assuming (1), it follows that the imaginary part of this metastable state energy be of the following form for small $|h|$:

$$\operatorname{Im} E(h\,e^{\pm i\pi}) = \mp\tilde{B}|h| \exp - \tilde{A}/|h|$$

with

$$\tilde{A} = A/K_2, \qquad \tilde{B} = BK_2. \tag{4}$$

In the work of Hamer (1983), metastable state energies were computed for complex values of h on finite lattices and extrapolated to the infinite system. The results were consistent with a singularity of the form (4) and numerical values of $\tilde{A}$ and $\tilde{B}$ at various subcritical couplings were obtained. We shall proceed to calculate an expression for $\tilde{A}$ using the droplet theory of I. This yields the following expression for A in two dimensions (equation 7(a) of I)

$$A = (\beta\Sigma)^2/8M, \tag{5}$$

M is the magnetisation while Σ is the surface free energy of an equilibrium Ising droplet of unit volume. Σ is given by

$$\Sigma = 2W^{1/2} \tag{6}$$

where W is the area of the Wulff plot, whose equation for the square lattice with anisotropic couplings is

$$h_1 \cosh \beta x + h_2 \cosh \beta y = 1$$

with

$$h_1 = \tanh(2K_2)/\cosh(2K_1) \qquad h_2 = \tanh(2K_1)/\cosh(2K_2). \tag{7}$$

Equations (6) and (7) have been taken from Zia and Avron (1982). The magnetisation M is given by (Chang 1952)

$$M = [1 - \operatorname{cosech}^2(2K_1)\operatorname{cosech}^2(2K_2)]^{1/8}. \tag{8}$$

Taking the limits $K_1 \to \infty$, $K_2 \to 0$, using (2), (4), (5)–(8) and writing W as an integral over x (and making the substitution $u = \sinh(\beta x/2)$) we finally obtain

$$\tilde{A} = 8(1-\mu)^{7/8} \int_0^1 \frac{(1-u^2)^{1/2}\,\mathrm{d}u}{(a^2+u^2)^{1/2}}$$

with

$$a^2 = 4\mu/(1-\mu)^2. \tag{9}$$

Values of $\tilde{A}$ computed from (9) are compared with Hamer's (1983) estimates in figure 1. The agreement is very good and, therefore, in strong support of the droplet picture which leads to the form (4). Finally we remark that expression (9) exhibits the expected divergence of $\tilde{A}$ as $\mu \to 0$ and vanishing at criticality, i.e.

$$\begin{aligned} \tilde{A} &\sim -4 \ln \mu, && \mu \to 0 \\ &\sim \pi(1-\mu)^{15/8}, && \mu \to 1. \end{aligned}$$

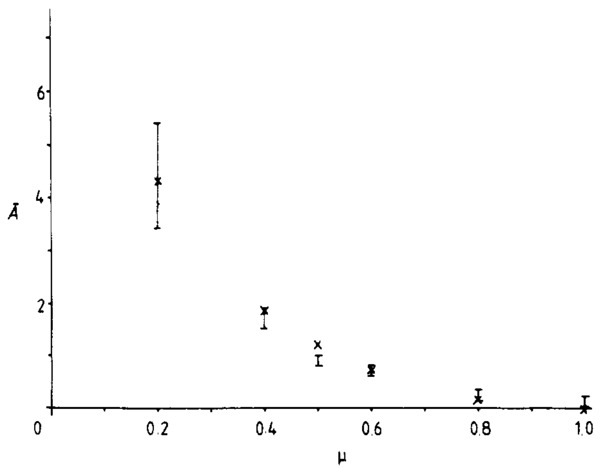

Figure 1. $\tilde{A}$ (see equation (4)) against coupling μ. Comparison of present theory (crosses) with numerical work (Hamer 1983, error bars).

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